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# Lyapunov Function Construction using Positive Dimensional Polynomial System Solver

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## Introduction

In the study and design of control systems, consideration of dynamical system stability is crucial. It is simple to confirm the stability of equilibria for linear systems. It is more difficult to demonstrate the stability of nonlinear system equilibria for nonlinear dynamical systems than it is for linear systems. The stability may be assessed using the Lyapunov function at the equilibrium.

The fundamental issue is how to determine the Lyapunov function at equilibrium for an autonomous polynomial system of differential equations. The Lyapunov function computation problem was changed into a quantifier elimination problem in [1, 2]. The method's drawback is that it has a computation complexity for quantifier elimination that is twice as complicated as the total number of variables. She et al[3] .'s symbolic solution to this issue first builds a specific semialgebraic system utilizing the local features of a Lyapunov function and its derivative, then uses CAD, a technique initially proposed by Collins in [4], to solve these inequalities. Semidefinite programming is used in the approach in [5] to look for the Lyapunov function. There are other algorithms as well.

In this study, we assume that the Lyapunov function has a quadratic shape and that certain of its coefficients are unknowable. Using the method described in [3], a few positive polynomials are first created, and then a positive dimensional polynomial system is built by including a few extra variables. By utilizing a numerical approach to solve the real root of the positive dimensional system, the parameter in the Lyapunov function is calculated.

The rest of this paper is organized as follows: Definitions and preliminaries about the Lyapunov function and the asymptotic stability analysis of differential system are given in Section 2. Section 3 reviews some methods for solving the real root of positive dimensional polynomial system.The new algorithm to compute the Lyapunov function and

some experiments are shown in Section 4. In Section 5, some examples are given to illustrates the efficiency of our algorithm. Finally, Section 6 draws a conclusion of this paper.

Stability Analysis of Differential Equations

In this section, some preliminaries on the stability analysis of differential equations are presented.

In this paper, we consider the following differential equations:

$$\begin{aligned}
\dot{x}_1 &= f_1(\mathbf{x}) \\
\dot{x}_2 &= f_2(\mathbf{x}) \\
&\vdots \\
\dot{x}_n &= f_n(\mathbf{x}),
\end{aligned}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $f_i \in \mathbb{R}[\mathbf{x}]$, and $x_i = x_i(t)$, $\dot{x}_i = dx_i/dt$. A point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ in the $n$-dimensional real Euclidean space $\mathbb{R}^n$ is called an *equilibrium* of differential system (1) if $f_i(\overline{\mathbf{x}}) = 0$ for all $i \in \{1, 2, \ldots, n\}$. Without loss of generality, we suppose the origin is an *equilibrium* of the given system in this paper.

In general, there exists two techniques to analyze the stability of an equilibrium: the Lyapunov's first method with the technique of linearization which considers the eigenvalues of the Jacobian matrix at equilibrium.

Theorem 1. Let $JF(x)$ denote the Jacobian matrix of system $\{f1,...,\}$ at point x. If all the eigenvalues of $JF(x)$ have negative real parts, then x is asymptotically stable. If the matrix $JF(x)$ has at least one eigenvalue with positive real part, then x is unstable.

For a small system, it is easy to obtain the eigenvalues of the matrix $JF(x)$; then one can analyze the stability of the equilibrium usingTheorem 1. For a high-dimensional system, solving the characteristic polynomial to get the exact zeros is a difficult problem. Indeed, to answer the question on stability of an equilibrium, we only need to know whether all the eigenvalues have negative real parts or not. Therefore, the theorem of Routh-Hurwitz [8] serves to determine whether all the roots of a polynomial have negative real parts.

Another method to determine asymptotic stability is to check if there exists a Lyapunov function at the point x, which is defined in the following.

Definition 2. Given a differential system and a neighborhood U of the equilibrium, a Lyapunov function with respect to the differential system is a continuously differential function : U → R such that

$$(1): F(\mathbf{0}) = 0 \text{ and } F(\overline{\mathbf{x}}) > 0 \text{ whenever } \overline{\mathbf{x}} \neq \mathbf{0};$$
$$(2): (d/dt)F(\mathbf{0}) = 0 \text{ and } (d/dt)F(\overline{\mathbf{x}}) < 0 \text{ whenever } \overline{\mathbf{x}} \neq \mathbf{0}.$$

**Solving the Real Roots of Positive Dimensional Polynomial System**

Solving polynomial system has been one of the central topics in computer algebra. It is required and used in many scientific and engineering applications. Indeed, we only care about the real roots of a polynomial system arising from many practical problems. For zero dimensional system, homotopy continuation method [9, 10] is a global convergence algorithm. For positive dimensional system, computing real roots of this system is a difficult and extremely important problem.

Due to the importance of this problem, many approaches have been proposed. The most popular algorithm which solves this problem is CAD; another is the so-called critical point methods, such as Seidenberg's approach of computing critical points of the distance function [11]. The algorithm in [12] uses the idea of Seidenberg to compute the real root of a positive dimensional defined by a signal polynomial; and extends it to a random polynomial system in [13]. Actually, these algorithms depend on symbolic computations, so they are restricted to small size systems because of the high complexity of the symbolic computation. In order to avoid this problem, homotopy method has been used to compute real root of polynomial system in [14, 15].

Recently, Wu and Reid [16] propose a new approach, which is different from the critical point technique. In order to facilitate the description of this algorithm, we suppose polynomial system $g = \{g1, g2,...,\}$; the system has $k$ polynomials, $n$ variables, and $k$

Theorem 3 (see [17]). Let $f(x) : Rn \to Rn$ be a polynomial system, and $x \in Rn$ . Let IR be the set of real intervals, and IR$n$ and IR$n\times n$ be the set of real interval vectors and real interval matrices, respectively. Given $X \in IRn$ with $0 \in X$ and $M \in IRn\times n$ satisfies $\nabla fi(x + X) \subseteq Mi$, for $i = 1, 2, . . . , n$. Denote by $In$ the identity matrix and assume

$$-F_{\mathbf{x}}^{-1}(\overline{\mathbf{x}})\, F(\overline{\mathbf{x}}) + \left(I_n - F_{\mathbf{x}}(\overline{\mathbf{x}})\, M\right) \quad \mathbf{X} \subseteq \operatorname{int}(\mathbf{X}),$$

where $Fx(x)$ is the Jacobian matrix of $F(x)$ at x. Then there is a unique $\hat{x} \in X$ such that (x)=0. Moreover, every matrix $M \in M$is nonsingular, and the Jacobian matrix $Fx(x)$is nonsingular

There may exist some components which have no intersection with these random hyperplanes. Some points on these components must be the solutions of the Lagrange optimization problem:

$$f = 0, \quad \sum_{i=1}^{k} \lambda_i \nabla f_i = \mathbf{n}.$$

Here n is a random vector in R . The system has $n+k$ equations and $n+k$ variables; thus we can find real points through solving system (3).

**Algorithm for Computing the Lyapunov Function**

In this section, we will present an algorithm for constructing the Lyapunov function. Our idea is to compute positive polynomial system which satisfies the definition of Lyapunov function first.

Then we solve the polynomial system deduced from the positive polynomial system using homotopy algorithm; at this step, we use the famous package hom4ps2 [18].

Given a quadratic polynomial $F(x)$, the following theorem gives a sufficient condition for the polynomial to be a Lyapunov function.

Theorem 4 (see [3]). Let $F(x)$ be a quadratic polynomial, for a given differential system; if $F(x)$ satisfies the fact that $He(F)|x=0$ is positive definite and $Hess((d/dt)F)|x=0$ is negative definite, then $F(x)$ is a Lyapunov function.

By the theory of linear algebra, one knows that the symmetric matrix $He(F)|x=0$ is positive definite if and only if all its eigenvalues are positive, and $Hess((d/dt)F)|x=0$ is negative definite if and only if all its eigenvalues are negative.

$$h = s^n + t_{n-1}s^{n-1} + \cdots + t_0$$

be a characteristic polynomial of a matrix; the following theorem deduced from the Descartes' rule of signs [19] can be used to determine whether $h$ has only positive roots or not.

Theorem 5 (see [3]). Suppose all the roots of a real polynomial $h$ are real; then its roots are all positive if and only if for all $1 \le i \le n$, $(-1)^i t_{n-i} > 0$.

Combine Theorems 4 and 5, finding that the Lyapunov function in quadratic form can be converted into solving the real root of some positive polynomial system, denoting it by

$$\text{Inequ} = \{g_1 > 0,\ g_2 > 0, \ldots, g_n > 0\}.$$

Suppose we have obtained the positive polynomial system as in (5), and denote the variable in the system by a. In order to obtain one value of a using numerical technique, we first convert the positive equation into equation. A simple ideal is to add new variable set x = $(x1, x2,...,)$, and construct the equation system as follows:

$$ps = \{g_1 - x_1^2, g_2 - x_2^2, \ldots, g_n - x_n^2\}.$$

If we find one real point (a, x) of system (6) such that there has nonzero element in x, then it is easy to see that the point a satisfies

$$\{g_1(\bar{\mathrm{a}}) > 0, g_2(\bar{\mathrm{a}}) > 0, \ldots, g_n(\bar{\mathrm{a}}) > 0\},$$

which means the differential system exists a Lyapunov function at the equilibrium.

Note that the number of variable is more than the number of equation in system (6); then the system $ps$ must be a positive dimensional polynomial system.

Recall the algorithm mentioned in Section 3; all of the algorithms obtain at least one real point in each connect component, and they use Theorem 3 to verify the existence of real root which deduces the low efficiency. However, in this paper, we only need one real point of system (6) to ensure the establishment of these inequalities in (7), so we verify the establishment of these inequalities using the residue of inequalities at the real part of every approximate real root of the system (6).

In the following we propose an algorithm to determine if there exists a Lyapunov function at the equilibrium.

Algorithm 6. Input: a differential system as defined in (1) and a tolerance $\epsilon$.

Output: a Lyapunov function or UNKNOW.

(1) Construct the positive polynomial.

(2) Convert the positive polynomial system into positive dimensional system defined in system (6).

(3) We choose $n$ random point $(\hat{x}1, \hat{x}2,...,\hat{x}n)$ and $n$ random vector k1, k2,..., k$n$; then construct $n$ hyperplane in R$n$ through $\hat{x}i$ with normal k$i$ for $i$ = 1, 2, . . . , $n$. Denote the set of this hyperplane by $ps2$.

(4) Let $ps = \{ps1, ps2\}$, and solve the square system using homotopy continuation algorithm, denoting solution of $ps$ by $roots$.

(5) for $s = 1 : length(roots)$

(a) if the norm of imaginary part of $roots\{s\}$ is smaller than $\epsilon$, then substitute the real part of $roots\{s\}$ into $\{g1,...,gn\}$, and denote the value by {V1, V2,..., V$n$}. If V$i$ > 0 for all $i \in \{1, 2, . . . , n\}$, then return the real part of $roots\{s\}$ and break the program.

(6) End for.

(7) Construct polynomial system $ps3 = \sum n\ i{=}1\ \lambda i \nabla f i$ = k, where $\lambda i$ is new variable and k are chosen from {k1,..., k$n$} randomly.

(8) Solve $\{ps1, ps3\}$ using homotopy continuation algorithm, denote its solution by $roots$, and go to Step 4.

(9) return UNKNOW.

In the following, we present a simple example to illustrate our algorithm.

Example 7. This is an example from [20]

$x = -x + 2y\dot{}3 - 2y^4$

$y = -x - y + xy.\dot{}$

Let Lyapunov function $(x, y) = x2 + axy + by^2$.

Step 1. We obtain the positive polynomial using Theorems 4 and 5 as follows:

$[2b + 2 > 0, -a2 + 4b > 0,$

$2a + 4b + 4 > 0, 4a2 + 4b2 - 16b > 0]$ .

Step 2. Convert system (9) into the following system:

$$ps_1 = \begin{cases} 2b + 2 - x_1^2 = 0 \\ -a^2 + 4b - x_2^2 = 0 \\ 2a + 4b + 4 - x_3^2 = 0 \\ 4a^2 + 4b^2 - 16b - x_4^2 = 0. \end{cases}$$

Step 3. Construct two hyperplanes $\{h1, h2\}$ in R6 randomly, where

$h1{=}0.09713178123584754a+ 0.04617139063115394b + 0.27692298496089x1 + 0.8234578283272926x2 + 0.694828622975817x3 + 0.3170994800608605x4 + 0.9502220488383549,$

$h2 = 0.3815584570930084a + 0.4387443596563982b + 0.03444608050290876x1 + 0.7655167881490024x2 + 0.7951999011370632x3 + 0.1868726045543786x4 + 0.4897643957882311.$

Step 4. Compute the roots of the augmented system $\{ps1 = 0, h1 = 0, h2 = 0\}$ using homotopy method, and we find the system has only 16 roots.

Step 5. We obtain the first approximate real root of the system

x = [−2.407604610156789, 4.633115716668555, 3.356520733339377, 3.568739680591174, −4.209186815331512, −5.909266734956268].

Substituting $a = -2.407604610156789$, $b = 4.633115716668555$ into the left of the positive polynomial in (9), we obtain the following result: [11.26623143, 12.73590291, 17.71725365, 34.91943333].

This ensure the establishment of inequality in (9). Thus, $F(x, y) = x2 + 4.633115716668555y^2 - 2.407604610156789xy$

is a Lyapunov function. If the random hyperplanes $\{h1, h2\}$ are as follows: $h1 = -3a - b + x1 + 2x2 - 2x3 - 2x4 - 3$, $h2 = 3a - 3b - x1 - 2x2 + x3 + 2x4 - 2$,

we find that polynomial system $\{h1 = 0, h2 = 0, ps = 0\}$ has no real root; then we go to Step 7 in Algorithm 6 and obtain the following system:

$$ps_3 = \begin{cases} -2\lambda_2 a + 2\lambda_3 + 8\lambda_4 a - 1 = 0 \\ 2\lambda_1 + 4\lambda_2 + 4\lambda_3 + \lambda_4(8b - 16) - 3 = 0 \\ -2\lambda_1 x_1 + 1 = 0 \\ -2\lambda_2 x_2 + 2 = 0 \\ -2\lambda_3 x_3 - 2 = 0 \\ -2\lambda_4 x_4 - 3 = 0. \end{cases}$$

Solving the system $\{ps1 = 0, ps3 = 0\}$, we find the first approximate real root and substitute the value of $a = 1.3053335232048229$, $b = 0.4314538107033688$ into the left of the positive polynomial in (9) and we obtain the following result:

[2.862907621406738, 0.021919636011159, 8.336482289223121, 0.656931019037197] .

This ensures the establishment of inequality in (9). Thus, $F(x, y) = x2 + 0.4314538107033688y2 + 1.3053335232048229xy$ is a Lyapunov function.

## Conclusion

We describe a numerical approach to compute the Lyapunov function at equilibria for a differential system based on the real root computation of positive dimensional polynomial system. We divide the method into two parts because we only require one real root of the positive dimensional system, as determined by the connection between the Lyapunov function and the positive dimensional system. Instead of utilizing the interval Newton's technique to check for the existence of the real root at each step, we utilize the positive polynomial system's residue at the approximate real root instead.

## References

[1] T. V. Nguyen, T. Mori, and Y. Mori, "Existence conditions of a common quadratic Lyapunov function for a set of second-order systems," Transactions of the Society of Instrument and Control Engineers, vol. 42, no. 3, pp. 241–246, 2006.

[2] T. V. Nguyen, T. Mori, and Y. Mori, "Relations between common Lyapunov functions of quadratic and infinity-norm forms for a set of discrete-time LTI systems," IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, vol. E89-A, no. 6, pp. 1794–1798, 2006.

[3] Z. She, B. Xia, R. Xiao, and Z. Zheng, "A semi-algebraic approach for asymptotic stability analysis," Nonlinear Analysis: Hybrid Systems, vol. 3, no. 4, pp. 588–596, 2009.

[4] G. E. Collins, "Quantifier elimination for real closed fields by cylindrical algebraic decomposition," in Automata Theory and Formal Languages, vol. 33 of Lecture Notes in Computer Science, pp. 134–183, Springer, Berlin, Germany, 1975.

[5] M. Bakonyi and K. N. Stovall, "Stability of dynamical systems via semidefinite programming," in Recent Advances in Matrix and Operator Theory, vol. 179 of Operator Theory: Advances and Applications, pp. 25–34, Birkhauser, Basel, Switzerland, 2008. ¨

[6] K. Forsman, "Construction of lyapunov functions using Groener bases," in ¨ Proceedings of the 30th IEEE Conference on Decision and Control, vol. 1, pp. 798–799, 1991.

[7] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in Proceedings of the 41st IEEE Conference on on Decision and Control, vol. 3, pp. 3482–3487, 2002.

[8] M. W. Hirsch and S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, vol. 60, Academic Press, New York, NY, USA, 1974.

[9] T. Y. Li, "Numerical solution of polynomial systems by homotopy continuation methods," Handbook of Numerical Analysis, vol. 11, pp. 209–304, 2003.

[10] A. J. Sommese and C. W. Wampler II, The Numerical Solution of Systems of Polynomials: Arising in Engineering and Science, World Scientific, Singapore, 2005.

[11] A. Seidenberg, "A new decision method for elementary algebra," Annals of Mathematics, vol. 60, no. 2, pp. 365–374, 1954.

[12] F. Rouillier, M.-F. Roy, and M. Safey El Din, "Finding at least one point in each connected component of a real algebraic set defined by a single equation," Journal of Complexity, vol. 16, no. 4, pp. 716–750, 2000.

[13] P. Aubry, F. Rouillier, and M. Safey El Din, "Real solving for positive dimensional systems," Journal of Symbolic Computation, vol. 34, no. 6, pp. 543–560, 2002.

[14] J. D. Hauenstein, "Numerically computing real points on algebraic sets," Acta Applicandae Mathematicae, vol. 125, no. 1, pp. 105–119, 2013.

[15] G. M. Besana, S. di Rocco, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler, "Cell decomposition of almost smooth real algebraic surfaces," Numerical Algorithms, vol. 63, no. 4, pp. 645–678, 2013.

[16] W. Wu and G. Reid, "Finding points on real solution componetns and applications to differential polynomial systems," in Proceedings of

the 38th International Symposium on Symbolic and Algebraic Computation, pp. 339–346, 2013.

[17] S. M. Rump and S. Graillat, “Verified error bounds for multiple roots of systems of nonlinear equations,” Numerical Algorithms, vol. 54, no. 3, pp. 359–377, 2010.

[18] T. Y. Li, HOM4PS-2.0, 2008, http://www.math.nsysu.edu.tw/~ leetsung/works/HOM4PS soft.htm.

[19] D. Wang and B. Xia, Computer Algebra, Tsinghua University Press, Beijing, China, 2004.

[20] S. H. Strogatz, Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering, Westview Press, 2001